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ESTIMATION OF BONFERRONI AND TOTAL TIME ON TEST CURVE USING OPTIMALLY SELECTED ORDER STATISTICS IN LARGE SAMPLES

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SUMMARY

Let X be a non-negative r.v. with continuous cdf $F_0\left(\frac{x-\mu}{\sigma}\right)$, where μ and σ are the location and scale parameters respectively. This paper deals with the estimation of the Bonferroni and Total Time on Test (TTT) function defined as $B_p(\mu,\sigma) = \frac{1}{p[\mu+\sigma E(z)]}\left\{p\mu + \sigma\int_0^p Q_0(t)dt\right\}$ and $T_p(\mu,\sigma) = pB_p(\mu,\sigma) + (1-p)\frac{\mu+\sigma Q_0(p)}{\mu+\sigma E(z)}$, respectively for $0 \leq p \leq 1$, ($Q_0(t)$ is the quantile function of $F_0(z)$) based on a few optimally selected order statistics when the sample size n is large. We use the asymptotically best linear unbiased estimators (ABLUE) of (μ,σ) based on an arbitrary set of $k \leq n$ order statistics $(x_{(r_1)}, x_{(r_2)}, \cdots, x_{(r_k)})$ with ranks $(r_1,\cdots,r_k)$ satisfying $1 \leq r_1 \leq r_2 \leq \cdots \leq r_k \leq n$ to define the estimate of the Bonferroni and TTT function and study the asymptotic efficiency property of these estimates. We propose how to obtain the optimum order statistics maximising asymptotic relative efficiency (ARE) of the estimator (relative to complete sample estimates). General theory as well as specific case studies are given for the Gamma, Exponential and Pareto distributions.

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1 Introduction

In the quest of measure of poverty and wealth, and income inequality, apart from Lorenz Curve, Bonferroni (1930) proposed a measure B_p which has suitable properties for studying wealth and income inequality. Compared with analogous indices such as Lorenz curve, it is also more sensitive to low levels of income distribution. This peculiarity makes the Bonferroni measure, B_p very useful in constructing poverty index. However, usefulness of the Bonferroni curve and Bonferroni index, B_p is to apply it to lifetesting and reliability (survival) studies. From practical point of view, one may consider a company who wants to evaluate the reliability of its own equipment in terms of component lifetimes to make replacement in adequate time. The analysis of life testing provides techniques which can also be used in longitudinal demographic surveys or in medical follow-ups where one can be interested in life time of a group of cancer patients, in the clinical conditions, undergoing the same medical treatments.

Let X be a non-negative random variable with continuous cdf $F_0\left(\frac{x-\mu}{\sigma}\right)$ and pdf $\frac{1}{\sigma}f_0\left(\frac{x-\mu}{\sigma}\right)$. Here μ and σ are the location and scale parameters of the distribution. Then the Bonferroni curve is defined by

$$B_p(\mu,\sigma) = \frac{1}{p[\mu+\sigma E(z)]}\left\{\mu + \sigma\int_0^p Q_0(t)dt\right\}$$

where $z = (x-\mu)/\sigma$ and $Q_0(t)$ is the quantile function of the cdf $F_0(z)$ defined by $Q_0(t) = \inf\{x : F(z) \geq t\}$. Since $\frac{\partial B_p}{\partial p} > 0$, the graph of $B_p(\mu,\sigma)$ is strictly increasing but we cannot say anything about the second derivative sign. Bonferroni curve may be convex or concave in some parts of the range of $p(0 < p < 1)$. Then the Bonferroni Index B_p is given by

$$B = 1 - \int_0^1 B_p(\mu,\sigma)dp.$$

Also, if $B_p^X(\mu,\sigma) > B_p^Y(\mu,\sigma)$, then the random variable X is stochastically larger than Y which orders the random variables X and Y.

Now, consider the TTT function, $T_p(\mu,\sigma)$ defined by

$$\begin{aligned} T_p(\mu,\sigma) &= pB_p(\mu,\sigma) + \frac{(1-p)}{\mu+\sigma E(z)}\{\mu+\sigma Q_0(p)\} \\ &= \frac{1}{(\mu+\sigma E(z))}\left[\mu + \sigma\left\{(1-p)Q_0(p) + \int_0^p Q_0(t)dt\right\}\right]. \end{aligned}$$

We may note that the Lorenz curve is given by

$$L_p(\mu,\sigma) = \frac{1}{\mu+\sigma E(z)}\left\{p\mu + \sigma\int_0^p Q_0(t)dt\right\}.$$

Clearly, the Bonferroni curve $B_p(\mu,\sigma)$ is always above the Lorenz curve and it is below the TTT curve $T_p(\mu,\sigma)$. Further, if $X \succeq Y$ i.e. $F_1(y) \geq F_1(x)$, then $B_p^Y(\mu,\sigma) \geq B_p^X(\mu,\sigma)$ and $T_p^Y(\mu,\sigma) \geq T_p^X(\mu,\sigma)$ which means that $F_2(y)$ is more IHR (increasing hazard rate) or less DHR (decreasing hazard rate) than $F_1(x)$. Thus, estimation of $B_p(\mu,\sigma)$ and $T_p^X(\mu,\sigma)$ served as the characteristics of the failure distributions when comparisons are made of two or more failure distributions. In this regard readers are referred to Bonferroni (1930), Giorgi (1998), Chandra and Singapurwalla (1981), Klefsjo (1984), Pham and Turkkan (1994) and Lorenz (1905) among others.

Let $X_1, \cdots, X_n$ be a sample of size n from $F_0\left(\frac{x-\mu}{\sigma}\right)$. Our problem is to estimate $B_p(\mu,\sigma)$ and $T_p(\mu,\sigma)$ based on a few selected k $(\leq n)$ sample quantiles $(x_{(r_1)}, x_{(r_2)}, \cdots, x_{(r_k)})$, where $r_i = [n\lambda_i]+1$, $i = 1, \cdots, k$ and $\lambda_1, \lambda_2, \cdots, \lambda_k$ are given fixed spacing satisfying $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < 1$. Parameter estimation based on a few selected sample quantiles was initiated by Ogawa (1951) and expanded by Saleh (1966) and Hassanein (1968, 71,77) among others. Not much work have been done on the problem of estimating the Bonferroni function $B_p(\mu,\sigma)$ and TTT function $T_p(\mu,\sigma)$ which is used in economic measures of inequality of income or variability in income distribution. In survival analysis and reliability one may use the two functions to measure the inequality of survival times or reliability of component equipment or variability of survival time or reliability.

These two functions may bring ordering to compare the amounts of inequality in two or more survival distributions.

In this paper, we present in Section 2 asymptotic theory to obtain asymptotically best linear unbiased estimates (ABLUE) of μ and σ of $F_0\left(\frac{x-\mu}{\sigma}\right)$. Not much work has been done on the problem of estimating the Bonferroni and TTT functions based on a few selected order statistics from a large sample. This problem of estimation of these functions based upon a fraction of the sample, say, the observations $(x_{(r_1)}, x_{(r_2)}, \cdots, x_{(r_k)})$ is relevant to these functions analysis. One may use the optimum estimators of μ and σ, say μ^* and σ^* based on a subset of order statistics and consider $B_p(\mu^*, \sigma^*)$ or $T_p(\mu^*, \sigma^*)$ as an estimate for $B_p(\mu, \sigma)$ or $T_p(\mu, \sigma)$ but not much is known about the properties of such an estimator. This paper is concerned with the theoretical justification, behaviour and properties of such estimators when the sample size n is large. In this paper, the theory will be utilized to study the properties of estimates of the two functions for the two-parameter Exponential, Pareto and Gamma distributions.

2 Asymptotic Theory of Least Square Estimation

Let $(x_1, x_2, \cdots, x_n)$ be a random sample from a cdf $F_0\left(\frac{x-\mu}{\sigma}\right)$ with pdf $\frac{1}{\sigma} f_0\left(\frac{x-\mu}{\sigma}\right)$ where μ and σ are unknown location and scale parameters respectively.
Let $x_{(1)}, x_{(2)}, \cdots, x_{(n)})$ be the order statistics corresponding to the sample and let $(x_{(r_1)}, x_{(r_2)}, \cdots, x_{(r_k)})$ be the $k(\leq n)$ order statistics with ranks $r_1, r_2, \cdots, r_k$ satisfying the relation $r_j = [n\lambda_j] + 1$, $j = 1, \cdots, k$, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < 1$, where [ ] is the Euler's notation for the integer contained in [ ]. Mosteller (1946) showed that $(x_{(r_1)}, x_{(r_2)}, \cdots, x_{(r_k)})'$ follows a k-vector normal distribution with mean vector $(\mu + \sigma u_1, \mu + \sigma u_2, \cdots, \mu + \sigma u_k)'$ and covariance matrix $\frac{\sigma^2}{n} V$ where

$$\begin{aligned} V &= \left(\left(v_{r_i r_j}\right)\right), \\ u_j &= inf\left\{z : F_0(z) \geq \lambda_j\right\}, j = 1, 2, \cdots, k \\ v_{r_i r_j} &= \left(\lambda_i \wedge \lambda_j - \lambda_i \lambda_j\right)\left\{f_0\left[F_0^{-1}(\lambda_i)\right] f_0\left[F_0^{-1}(\lambda_j)\right]\right\}^{-1} \end{aligned}$$

Using Gauss-Markov theorem, we obtain the ABLUE of μ and σ by minimizing

$$\left(X_{(.)} - \mu 1_k - \sigma \boldsymbol{u}\right)' V^{-1}\left(X_{(.)} - \mu 1_k - \sigma \boldsymbol{u}\right),$$

where $X_{(.)} = \left(X_{(r_1)}, X_{(r_2)}, \cdots, X_{(r_k)}\right)'$, $\boldsymbol{u} = (u_1, \cdots, u_k)'$, $1_k = (1, 1, \cdots, 1)'$.

With respect to μ and σ to obtain the normal equations

$$\begin{aligned} \mu^*\left(1'V^{-1}1\right) + \sigma^*\left(1'V^{-1}\boldsymbol{u}\right) &= 1'V^{-1}X_{(.)} \\ \mu^*\left(1'V^{-1}\boldsymbol{u}\right) + \sigma^*\left(\boldsymbol{u}'V^{-1}\boldsymbol{u}\right) &= \boldsymbol{u}'V^{-1}X_{(.)} \end{aligned}$$

Let $K_1 = 1'V^{-1}1$, $K_2 = \boldsymbol{u}'V^{-1}\boldsymbol{u}$, $K_3 = 1'V^{-1}\boldsymbol{u}$ and $\Delta = K_1K_2 - K_3^2$.

Then, the ABLUE of μ and σ are

$$\begin{aligned} \mu^* &= \frac{1}{\Delta}\left[\left(1'V^{-1}X_{(.)}\right)K_2 - \left(\boldsymbol{u}'V^{-1}X_{(.)}\right)K_3\right] \\ \sigma^* &= \frac{1}{\Delta}\left[\left(\boldsymbol{u}'V^{-1}X_{(.)}\right)K_1 - \left(1'V^{-1}X_{(.)}\right)K_3\right] \end{aligned}$$

Also K_1, K_2 and K_3 has the explicit forms given by

$$\begin{aligned} K_1 &= \sum_{i=1}^{k+1} \frac{(f_i - f_{i-1})^2}{\lambda_i - \lambda_{i-1}} \\ K_2 &= \sum_{i=1}^{k+1} \frac{(f_i u_i - f_{i-1} u_{i-1})^2}{\lambda_i - \lambda_{i-1}} \\ K_3 &= \sum_{i=1}^{k+1} \frac{(f_i - f_{i-1})(f_i u_i - f_{i-1} u_{i-1})}{\lambda_i - \lambda_{i-1}} \end{aligned}$$

where f_i is the density function at the λ_i-quantile i.e. $f_i = f(u_i)$, $f_0 = f_{k+1} = 0$, $\lambda_0 = 0$, $\lambda_{k+1} = 1$ and $\int_{\infty}^{u_i} f(x)dx = \lambda_i$.

Then one can varify (see Sarhan and Greenberg, 1962) that the covariance matrix for (μ^*, σ^*) is given by

$$\frac{\sigma^2}{n\Delta} \begin{bmatrix} K_2 & -K_3 \\ -K_3 & K_1 \end{bmatrix}.$$

Thus, $Var(\mu^*) = \frac{\sigma^2}{n\Delta} K_2$, $Var(\sigma^*) = \frac{\sigma^2}{n\Delta} K_1$ and $Cov(\mu^*, \sigma^*) = \frac{\sigma^2}{n\Delta} K_3$.

3 Estimation of a Function of Location and Scale Parameters

Now, we consider the point estimation of a function $g(\mu, \sigma)$ based on the sample quantiles $X_{(r_1)}, X_{(r_2)}, \cdots, X_{(r_k)}$. Let μ^* and σ^* be the ABLUE of μ and σ respectively based on $X_{(r_1)}, X_{(r_2)}, \cdots, X_{(r_k)}$.

Then, we propose $g(\mu^*, \sigma^*)$ as the point estimate of $g(\mu, \sigma)$ based on the given $k(\leq n)$ sample quantiles.

To obtain the asymptotic variance of $g(\mu^*, \sigma^*)$, we use the local linearization theorem as in Rao (1965; pp 321). We note that asymptotic joint distribution of

$$\left\{\sqrt{n}\,(\mu^* - \mu), \sqrt{n}\,(\mu^* - \mu)\right\}'$$

is a bivariate normal with mean $(0, 0)$ and variance covariance matrix given by

$$\frac{\sigma^2}{n\Delta} \begin{bmatrix} K_2 & -K_3 \\ -K_3 & K_1 \end{bmatrix}$$

Thus, using Rao (1965) we have

$$\begin{aligned} \text{AsVar} g(\mu^*, \sigma^*) &= \text{AsVar}(\mu^*) \left(\frac{\partial g(\mu, \sigma)}{\partial \mu}\right)^2 + \text{AsVar}(\sigma^*) \left(\frac{\partial g(\mu, \sigma)}{\partial \sigma}\right)^2 \\ &\quad + \text{AsCov}(\mu^*, \sigma^*) \left(\frac{\partial g(\mu, \sigma)}{\partial \mu}\right) \left(\frac{\partial g(\mu, \sigma)}{\partial \sigma}\right) \end{aligned} \tag{3.1}$$

evaluated at $(\mu^* = \mu, \sigma^* = \sigma)$.

Upon simplification (3.1) yields

$$\text{AsVar}g(\mu^*, \sigma^*) = \left[\omega_1^2 K_2 + \omega_2^2 K_2 - 2\omega_1\omega_2 K_3\right]/n\Delta,$$

where $\omega_1 = \frac{\partial g(\mu,\sigma)}{\partial \mu}$ and $\omega_2 = \frac{\partial g(\mu,\sigma)}{\partial \sigma}$.

Let $(\bar{\mu}, \bar{\sigma})'$ be the BLUE of MLE of $(\mu, \sigma)'$ using all the observations. Let $\bar{g}(\bar{\mu}, \bar{\sigma})$ be the point estimate of $g(\mu, \sigma)$. Then we obtain

$$\text{AsVar}\bar{g}(\bar{\mu}, \bar{\sigma}) = \left[\omega_1^2 I_{22} + \omega_2^2 I_{11} - 2\omega_1\omega_2 I_{12}\right]/n|I|,$$

where

$$I = \begin{bmatrix} I_{11} & I_{12} \\ I_{12} & I22 \end{bmatrix}$$

is the Fisher's information matrix for (μ, σ).

Therefore, the asymptotic relative efficiency (ARE) of $g(\mu^*, \sigma^*)$ relative to $\bar{g}(\bar{\mu}, \bar{\sigma})$ is given by

$$\text{ARE}\,[g : \bar{g}] = \frac{\omega' \mathbf{I}^{-1} \omega}{\omega' \mathbf{K}^{-1} \omega}, \quad \omega' = (\omega_1, \omega_2) \tag{3.2}$$

with

$$\mathbf{K} = \begin{bmatrix} K_1 & K_3 \\ K_3 & K_2 \end{bmatrix}.$$

To obtain the optimum spacings for the estimation of $g(\mu^*, \sigma^*)$, we maximize (3.2) with respect to $(u_1, u_2, \cdots, u_k)'$ which involves the unknown $\omega = (\omega_1, \omega_2)'$. As such, we apply the Courant-Fisher theorem (Rao, pp 48-53) and find the upper and lower bound of (3.2). Thus we have

$$\text{Ch}_{\min}(\mathbf{I}^{-1}\mathbf{K}) \leq \frac{\omega' \mathbf{I}^{-1} \omega}{\omega' \mathbf{K}^{-1} \omega} \leq \text{Ch}_{\max}(\mathbf{I}^{-1}\mathbf{K}),$$

where $\text{Ch}_{\min}(\text{A})$ and $\text{Ch}_{\max}(\text{A})$ are the minimum and maximum characteristics roots of A. It is well known that

$$\text{Ch}_{\min}(\mathbf{I}^{-1}\mathbf{K}) + \text{Ch}_{\max}(\mathbf{I}^{-1}\mathbf{K}) = \text{tr}(\mathbf{I}^{-1}\mathbf{K})$$
$$\text{Ch}_{\min}(\mathbf{I}^{-1}\mathbf{K}) \times \text{Ch}_{\max}(\mathbf{I}^{-1}\mathbf{K}) = \text{Det}(\mathbf{I}^{-1}\mathbf{K})$$

where tr(A) and Det(A) are the trace and determinant of the matrix A. Also

$$\frac{1}{2}\text{tr}(\mathbf{I}^{-1}\mathbf{K}) > \sqrt{\text{Ch}_{\min}(\mathbf{I}^{-1}\mathbf{K})\text{Ch}_{\max}(\mathbf{I}^{-1}\mathbf{K})}.$$

Thus, we maximize the average ARE(AARE) give by

$$\frac{1}{2}\text{tr}(\mathbf{I}^{-1}\mathbf{K}) = \frac{1}{2}\left\{(I_{22}K_1 + I_{11}K_2 - 2I_{12}K_3)\right\}/(I_{11}I_{22} - I_{12}^2)$$

with respect to $(u_1, u_2, \cdots, u_k)$.

Further

$$\begin{aligned} \left|\mathbf{I}^{-1}\mathbf{K}\right| &= \frac{\Delta}{|\mathbf{I}|} \\ \mathrm{Ch}_{\max}(\mathbf{I}^{-1}\mathbf{K}) &= \frac{1}{2}\mathrm{tr}(\mathbf{I}^{-1}\mathbf{K}) + \frac{1}{2}\left\{\mathrm{tr}^2(\mathbf{I}^{-1}\mathbf{K}) - 4|\mathbf{I}^{-1}\mathbf{K}|\right\}^{\frac{1}{2}} \end{aligned} \tag{3.3}$$

Thus, we may maximize (3.3) with respect to $\mathbf{u} = (u_1, u_2, \cdots, u_k)'$ to obtain the optimum spacings for the k-sample quantiles.

We now consider the details of the estimation of Bonferroni and TTT curve when the distributions are Gamma, Exponential and Pareto in the following sections.

4 Exponential Distribution

Let $x_1, \cdots, x_n$ be a random sample of size n from the exponential distribution

$$f(x) = \sigma^{-1} exp\left(-\frac{x-\mu}{\sigma}\right); \quad x \geq \mu, \ \sigma > 0$$

Then the Bonferroni function of order $p(0 < p < 1)$ is given by

$$B_p(\mu, \sigma) = 1 - \frac{\sigma}{\mu + \sigma}(1-p)\ln(1-p)^{-1}$$

Similarly, the TTT function is given by

$$T_p(\mu, \sigma) = 1 - \frac{(1-p)\sigma}{\mu + \sigma} - \frac{\sigma}{\mu + \sigma}(1-p)^2 \ln(1-p)^{-1}$$

Both the Bonferroni and the TTT functions depend on (μ, σ) of the exponential distribution. Our primary objective is the estimation of $B_p(\mu, \sigma)$ and $T_p(\mu, \sigma)$ based on say $k(1 \leq k \leq n)$ optimum order statistics. First we note that the MLE of $B_p(\mu, \sigma)$ and $T_p(\mu, \sigma)$ are given by

$$\begin{aligned} \bar{B}_p(\bar{\mu}, \bar{\sigma}) &= 1 - \frac{\bar{\sigma}}{\bar{\mu} + \bar{\sigma}}(1-p)\ln(1-p)^{-1} \\ \text{and} \quad \bar{T}_p(\bar{\mu}, \bar{\sigma}) &= 1 - \frac{(1-p)\bar{\sigma}}{\bar{\mu} + \bar{\sigma}} + \frac{\bar{\sigma}}{\bar{\mu} + \bar{\sigma}}(1-p)^2 \ln(1-p)^{-1} \end{aligned}$$

respectively, where

$$\bar{\mu} = \frac{nx_{(1)} - \bar{x}}{n-1} \quad \text{and} \quad \bar{\sigma} = \frac{n(\bar{x} - x_{(1)})}{n-1)}$$

with variance-covariance matrix

$$\frac{\sigma^2}{n}\begin{bmatrix} \frac{1}{n-1} & -\frac{1}{n-1} \\ -\frac{1}{n-1} & \frac{n}{n-1} \end{bmatrix}.$$

Now to estimate $B_p(\mu, \sigma)$ and $T_p(\mu, \sigma)$ curve based on an arbitrary set $k(2 \leq k \leq n)$ of order statistics we consider the order statistics determined by k real numbers $\lambda_1, \lambda_2, \cdots, \lambda_k$ satisfying the relation $0 < \lambda_1 < \cdots < \lambda_k < 1$ such that

$$n_i = [n\lambda_i] + 1, \ i = 1, 2, \cdots, k.$$

The joint distribution of $\left(x_{(n_1)}, \cdots, x_{(n_n)}\right)$ is k-variate normal with mean-vector $(\mu+\sigma u_1, \cdots, \mu+\sigma u_k)'$ and covariance matrix $\frac{\sigma^2}{n}V$, where $u_i = \ln(1-\lambda_i)^{-1},\ i = 1, 2, \cdots, k$ and $V = ((V_{ij})),\ v_{ii} = V_{ij} = V_{ji} = (e^{u_i} - 1),\ i < j,\ i, j = 1, 2, \cdots, k$.

The inverse of v is $V^{-1} = \left((V^{ij})\right)$, where

$$
\begin{aligned}
V^{i,i} &= e^{2u_i}\left(e^{-u_{i-1}} - e^{-u_{i+1}}\right) / \left(e^{-u_i} - e^{-u_{i+1}}\right)\left(e^{-u_{i+1}} - e^{-u_{i+2}}\right) \\
V^{i,i-1} &= V^{i-1,i} = 1/\left(e^{-u_i} - e^{-u_{i+1}}\right),\ V^{ij} = 0 \text{ for } \mid i-j \mid > 1
\end{aligned}
$$

Using the Gauss-Markov Theorem we minimize

$$
\left(\mathbf{X}_{(.)} - \mu \mathbf{1}_k - \sigma \mathbf{u}\right)' V^{-1} \left(\mathbf{X}_{(.)} - \mu \mathbf{1}_k - \sigma \mathbf{u}\right)
$$

with respect to μ and σ where $\mathbf{X}_{(.)} = \left(x_{(n_1)}, \cdots, x_{(n_k)}\right)'$, $\mathbf{1}_k = (1, \cdots, 1)'$ a k tuple of $\mathbf{1}$'s and $\mathbf{u} = (u_1, \cdots, u_k)'$ to obtain the asymptotically best linear unbiased estimators (ABLUE) of μ and σ as

$$
\begin{aligned}
\widehat{\mu} &= x_{(n_1)} - \widehat{\sigma} u_1 \\
\widehat{\sigma} &= b_1 x_{(n_1)} + \cdots + b_1 x_{(n_k)}
\end{aligned}
$$

where

$$
\begin{aligned}
b_1 &= \frac{1}{L}\left(\frac{u_2 - u_1}{e^{u_2} - e^{u_1}}\right),\ L = \sum_{i=2}^{k} \frac{(u_i - u_{i-1})}{e^{u_i} - e^{u_{i-1}}} \\
b_i &= \frac{1}{L}\left(\frac{u_i - u_{i-1}}{e^{u_i} - e^{u_{i-1}}} - \frac{u_{i+1} - u_i}{e^{u_{i+1}} - e^{u_i}}\right) \\
b_k &= \frac{1}{L}\left(\frac{u_k - u_{k-1}}{e^{u_k} - e^{u_{k-1}}}\right)
\end{aligned}
$$

The variance-covariance matrix of the ABLUE is

$$
\frac{\sigma^2}{n}\begin{bmatrix} \frac{u_1^2}{L} + (e^{u_1} - 1) & -\frac{u_1}{L} \\ -\frac{u_1}{L} & \frac{1}{L} \end{bmatrix}
$$

The inverse of this matrix is given by

$$
n\sigma^{-2}\begin{bmatrix} \frac{1}{e^{u_1}-1} & \frac{u_1}{e^{u_1}-1} \\ \frac{u_1}{e^{u_1}-1} & \frac{u_1}{e^{u_1}-1} + L \end{bmatrix} \tag{4.1}
$$

The matrix (4.1) will be needed later to compute the asymptotic relative efficiency (ARE) of the estimator of $B_p(\mu, \sigma)$ and $T_p(\mu, \sigma)$ based on the optimum k order statistics.

Thus, we propose the estimates of $B_p(\mu, \sigma)$ and $T_p(\mu, \sigma)$ as

$$
\begin{aligned}
\widehat{B}_p(\widehat{\mu}, \widehat{\sigma}) &= 1 - \frac{\widehat{\sigma}}{\widehat{\mu} + \widehat{\sigma}}(1-p)\ln(1-p)^{-1} \\
\text{and } \widehat{T}_p(\widehat{\mu}, \widehat{\sigma}) &= 1 - \frac{(1-p)\widehat{\sigma}}{\widehat{\mu} + \widehat{\sigma}} - \frac{\widehat{\sigma}}{\widehat{\mu} + \widehat{\sigma}}(1-p)^2\ln(1-p)^{-1}
\end{aligned}
$$

respectively.

The asymptotic variance of $\widehat{B}_p(\widehat{\mu}, \widehat{\sigma})$ and $\widehat{T}_p(\widehat{\mu}, \widehat{\sigma})$ may be obtained by the use of the local linearization theorem as in Rao (1965, pp 32). We note that the matrix $\mathbf{K}$ for this problem is given by

$$\mathbf{K} = \begin{bmatrix} \frac{u_1^2}{L} + (e^{u_1} - 1) & -\frac{u_1}{L} \\ -\frac{u_1}{L} & \frac{1}{L} \end{bmatrix}.$$

Thus, we maximize

$$\frac{u_1^2}{e^{u_1 - 1}} + L \tag{4.2}$$

for $(u_1, u_2, \cdots, u_k)$. Note that we can write (4.2) as

$$\frac{u_1^2}{e^{u_1} - 1} + e^{-u_1} Q_{k-1}, Q_{k-1} = \sum_{i=1}^{k} \frac{(t_i - t_{i-1})^2}{e^{t_i} - e^{t_{i-1}}}$$

with $t_0 = 0$ and $t_i = u_{i+1} - u_i$.

To maximize (4.2), we first consider the maximum of Q_{k-1} say Q_{k-1}^0. Then, we maximize (4.2) subject to $Q_{k-1} = Q_{k-1}^0$. Thus, we solve the equation in u_1 for fixed $Q_{k-1} = Q_{k-1}^0$.

The solution of u_1 is the root of

$$\frac{u_1^2}{e^{u_1} - 1} = 1 + \sqrt{1 - Q_{k-1}^0}, \tag{4.3}$$

where $p_1^0 = 1 - e^{-u_1^0}$ is the spacing of the smallest order statistic. The rank of this order statistic is

$$n_1^0 = [np_1^0] + 1$$

To obtain the remaining order statistics let $\lambda_1^0, \cdots, \lambda_{k-1}^0$ be the optimum spacing to obtain Q_{k-1}^0. These are optimum spacings to obtain the ABLUE of σ alone (Ogawa, 1951). Then, $(k-1)$ optimum spacings for $\widehat{L}_p(\widehat{\mu}, \widehat{\sigma})$ are given by

$$p_{i+1}^0 = p_i^0 + (1 - p_i^0)\lambda_i, \quad i = 1, 2, \cdots, k-1$$

Hence, the ranks are given by

$$n_i^0 = [np_i^0] + 1, \quad i = 1, 2, \cdots, k$$

The optimum ABLUE of μ and σ are

$$\widehat{\mu}^0 = x_{n_1^0} - \widehat{\sigma}^0 \ln\left(1 - p_1^0\right)^{-1}, \tag{4.4}$$

$$\widehat{\sigma}^0 = \sum b_j^0 x_{(n_j)}. \tag{4.5}$$

The coefficient $(b_1^0, \cdots, b_k^0)$ are obtained by (Ogawa, 1951). Thus the optimum Lorenz curve estimator is given by

$$\begin{aligned} \widehat{B}_p(\widehat{\mu}^0, \widehat{\sigma}^0) &= 1 - \frac{\widehat{\sigma}^0}{\widehat{\mu}^0 + \widehat{\sigma}^0}(1-p)\ln(1-p_1)^{-1} \\ \text{and } \widehat{T}_p(\widehat{\mu}^0, \widehat{\sigma}^0) &= 1 - \frac{(1-p)\widehat{\sigma}^0}{\widehat{\mu}^0 + \widehat{\sigma}^0} + \frac{\widehat{\sigma}^0}{\widehat{\mu}^0 + \widehat{\sigma}^0}(1-p)^2\ln(1-p)^{-1} \end{aligned}$$

with ARE given by

$$\text{ARE} = (1-p_1^0)^{-1}\left(1-p_1^0\right)\left\{\ln^2(1-p_1^0)/p_1^0 + Q_{k-1}^0\right\}.$$

4.1 Example 1.

Suppose we want to estimate $B_p(\mu, \sigma)$ based on 6 order statistics in a sample of size 80 from an exponential distribution. From Ogawa (1951) and Sarhan and Greenberg (1962), for $k = 5$, the optimum spacings for σ alone are $\lambda_1 = 0.3478$, $\lambda_2 = 0.6042$, $\lambda_3 = 0.7828$, $\lambda_4 = 0.8978$ and $\lambda_5 = 0.9631$ and optimum u_1 by solving (4.3) is $u_1^0 = 0.4274$. Thus, $p_1^0 = 1 - e^{-u_1^0} = .3478$, $p_2^0 = .5746$, $p_3^0 = .7419$, $p_4^0 = .8583$, $p_5^0 = .9333$ and $p_6^0 = .9759$. The optimum ranks are then $n_1^0 = 28$, $n_2^0 = 46$, $n_3^0 = 60$, $n_4^0 = 69$, $n_5^0 = 75$ and $n_6^0 = 79$. The coefficients b_j^0's are $b_1^0 = 0.3108$, $b_2^0 = 0.2228$, $b_3^0 = 0.1492$, $b_4^0 = 0.0902$, $b_5^0 = 0.0456$ and $b_6^0 = 0.0156$. The ABLUE of μ and σ are

$$\begin{aligned} \widehat{\mu}^0 &= x_{(28)} - 0.4274\widehat{\sigma}^0 \\ \widehat{\sigma} &= 0.3108x_{(28)} + 0.2228x_{(46)} + 0.1492x_{(69)} + 0.0902x_{(75)} + 0.0156x_{(79)}. \end{aligned}$$

The estimate of $B_p(\mu, \sigma)$ and $T_p(\mu^0, \sigma^0)$ based on $\widehat{\mu}^0$ and $\widehat{\sigma}^0$ is

$$\begin{aligned} \widehat{B}_p(\widehat{\mu}^0, \widehat{\sigma}^0) &= 1 - \frac{\widehat{\sigma}^0}{\widehat{\mu}^0 + \widehat{\sigma}^0}(1-p)\ln(1-p_1)^{-1} \\ \widehat{T}_p(\widehat{\mu}^0, \widehat{\sigma}^0) &= 1 - \frac{(1-p)\widehat{\sigma}^0}{\widehat{\mu}^0 + \widehat{\sigma}^0} + \frac{\widehat{\sigma}^0}{\widehat{\mu}^0 + \widehat{\sigma}^0}(1-p)^2\ln(1-p)^{-1}. \end{aligned}$$

The ARE of both of the estimators is 0.7892.

5 Pareto Distribution

In this section we consider the Benferroni and TTT curve corresponding to the Pareto distribution is generally used to study the income distribution. Thus, Bonferroni curve describes the disparity of income distribution in the case which is relevant to economic studies.

The Pareto distribution is given by the cdf

$$F(x, \theta, \gamma) = 1 - \left(\frac{\theta}{x}\right)^{\gamma}, \quad x \geq \theta, \quad \gamma > 0$$

The Bonferroni function of the distribution is given by

$$B_p(\theta, \gamma^{-1}) = \frac{1}{p}\left\{1 - (1-p)^{1-\gamma^{-1}}\right\}, \quad 0 \leq p \leq 1$$

depends only on γ. However, if we consider log-transformation, then $\log x$ is exponentially distributed with location parameter $\ln\theta$ and scale parameter γ^{-1}.

Then, consider the Bonferroni function using $\log x$ distribution, we then have

$$B_p(\ln\theta, \gamma^{-1}) = 1 - \frac{\gamma}{\gamma\ln\theta + 1}(1-p)\ln(1-p)^{-1} \tag{5.1}$$

Now, the estimates of $\mu = \ln\theta$ and $\gamma^{-1} = \sigma$ may be obtained using (4.4) and (4.5) as

$$\widehat{\mu} = \widehat{\ln\theta} = \ln x_{(n_1)} - \widehat{\sigma}u_1 \tag{5.2}$$

$$\text{and } \widehat{\sigma} = b_1 \ln x_{(n_1)} + \cdots + b_k \ln x_{(k)} \tag{5.3}$$

where $\widehat{\theta} = e^{\widehat{\mu}}$ and $\widehat{\gamma} = \widehat{\sigma}^{-1}$. The estimate of (5.1) is obtained by substituting (5.2) and (5.3) in (5.1). ARE expressions are the same as the exponential distribution.

6 Gamma Distribution

In this section, we consider the Bonferroni and TTT functions corresponding to the two-parameter Gamma distribution defined by

$$f(x \mid \mu, \sigma, \alpha) = \frac{1}{\sigma\Gamma(\alpha)}\left(\frac{x-\mu}{\sigma}\right)^{\alpha-1} exp\left\{-\left(\frac{x-\mu}{\sigma}\right)\right\} \tag{6.1}$$

Here, $x \geq \mu$, $\sigma > 0$ and $\alpha \geq 3$. The Fisher's Information matrix for this distribution is given by $\sigma^2 I$ where

$$I = \begin{bmatrix} \frac{1}{\alpha-2} & 1 \\ 1 & \alpha \end{bmatrix}.$$

and

$$I^{-1} = \frac{\alpha-2}{2}\begin{bmatrix} \alpha & -1 \\ -1 & \frac{1}{\alpha-2} \end{bmatrix}.$$

Let $\frac{x-\mu}{\sigma}$, then we have from (6.1)

$$f_0(z; 0, 1, \alpha) = \frac{1}{\Gamma(\alpha)} z^{\alpha-1}e^{-z}, \quad z \geq 0. \tag{6.2}$$

Let then $F_0(z; 0, 1, \alpha)$ be the cdf, consequently the quantile function of (6.2) is given by

$$Q_0(\lambda) = F_0^{-1}(z; 0, 1, \alpha) = \inf\{z : F_0(z; 0, 1, \alpha) > \lambda\}$$

and the Bonferroni and TT functions are then defined by

$$B_p(\mu, \sigma) = \frac{1}{p[\mu + \sigma E(z)]}\left\{p\mu + \sigma\int_0^p Q_0(t)dt\right\}$$

$$\text{and } T_p(\mu, \sigma) = pB_p(\mu, \sigma) + (1-p)\frac{\mu + \sigma Q_0(p)}{\mu + \sigma E(z)}$$

The estimate of $B_p(\mu,\sigma)$ is obtained by substituting the ABLUE of μ and σ, namely μ^* and σ^* based on k sample quantiles with ranks $n_1, n_2, \cdots, n_k$, where

$$n_j = [n\lambda_j] + 1, \quad j = 1, \cdots, k$$

and the estimates μ^* and σ^* are given by

$$\mu^* = \sum_{i=1}^{k} a_i X_{n_i}$$

and

$$\sigma^* = \sum_{i=1}^{k} b_i X_{n_i}$$

respectively.

The optimum ranks n_j are obtained by maximizing

$$\begin{aligned} tr\left(I^{-1}K\right) &= \frac{1}{2}\left[(\alpha-2)(\alpha K_1 - 2K_3 + K_2)\right] \\ \text{or} \quad Ch_{max} tr\left(I^{-1}K\right) &= \frac{1}{2}\left[(\alpha-2)(\alpha K_1 - 2K_3 + K_2)\right] + \frac{1}{4}\sqrt{C} \end{aligned}$$

where $C = (\alpha(\alpha-2)K_1 - K_2)^2 + (\alpha-2)(K_3-(\alpha-2)K_1)(\alpha K_3 - K_2)$. The values of $\lambda_1, \cdots, \lambda_k, K_1, K_2, K_3$ that maximize $tr\left(I^{-1}K\right)$ and $Ch_{max} tr\left(I^{-1}K\right)$ respectively for $\alpha = 3(1)10$ and $k = 2(1)10$ are given in a table which is available on request from the first author.

6.1 Example 2

Assume that we want to estimate $B_p(\mu,\sigma)$ based on 6 oredr statistics in a sample of size 1000 from a Gamma distribution. For $k = 6$, the optimum ranks for estimating μ and σ are $\lambda_1 = 0.0003$, $\lambda_2 = 0062$, $\lambda_3 = 0.0578$, $\lambda_4 = 0.3637$, $\lambda_5 = 0.7458$ and $\lambda_6 = 0.9432$. Also, optimum U_i's are $U_1 = 0.1250$, $U_2 = 0.3654$, $U_3 = 0.8689$, $U_4 = 2.1641$, $U_5 = 3.8933$ and $U_6 = 6.1198$. The coefficients for estimating μ are $a_1 = 0.2201$, $a_2 = 0.3905$, $a_3 = 0.4581$, $a_4 = 0.0964$, $a_5 = -0.1047$ and $a_6 = -0.0603$. The coefficients for estimating σ are $b_1 = -0.0748$, $b_2 = -0.1271$, $b_3 = -0.1040$, $b_4 = 0.1047$, $b_5 = 0.1399$ and $b_6 = 0.0612$. The optimum ranks are then $n_1^0 = 1$, $n_2^0 = 7$, $n_3^0 = 58$, $n_4^0 = 368$, $n_5^0 = 746$ and $n_6^0 = 944$.

The ABLUE for μ and σ are respectively $\mu^* = .2201x_{(1)} + .3905x_{(7)} + .4581x_{(58)} + .0964x_{(368)} - .1047x_{(746)} - .0603x_{(944)}$ and $\sigma^* = -0.48x_{(1)} - .1271x_{(7)} - .1040x_{(58)} + .1047x_{(368)} + .1399x_{(746)} + .0612x_{(944)}$. The estimators based on μ^* and σ^* of the Bonferroni and TTT functions are given by

$$\begin{aligned} B_p^*(\mu,\sigma) &= \frac{1}{p[\mu^* + \sigma^* E(z)]}\left\{p\mu^* + \sigma^* \int_0^p Q_0(t)dt\right\} \\ \text{and} \quad T_p(\mu^*,\sigma^*) &= pB_p(\mu^*,\sigma^*) + (1-p)\frac{\mu^* + \sigma^* Q_0(p)}{\mu^* + \sigma^* E(z)} \end{aligned}$$

respectively.

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