Journal of Statistical Research *ISSN 0256 - 422 X*
2003, Vol. 37, No. 1, pp. 17-19
Bangladesh

ON THE PROPRIETY OF POSTERIORS FOR PROPORTIONAL HAZARDS MODELS

Gauri S. Datta
University of Georgia

Malay Ghosh
University of Florida
Email: ghoshm@stat.ufl.edu

Bani K. Mallick
Texas A & M University

SUMMARY

The paper considers the propriety of posteriors for proportional hazards models with exponential baseline hazards.

Keywords: Baseline hazard, exponential, gamma prior, simple regression, uniform prior.

AMS Classification: 62C10

1 Introduction

Recent years have witnessed a surge of interest in the Bayesian analysis of Cox's (1972) proportional hazards models. A very succinct account of the same is given in Ibrahim, Chen and Sinha (2001). Although much of the Bayesian literature is devoted to semiparametric inference for these models, some authors (e.g. Dellaportas and Smith, 1993) have considered fully parametric Bayesian analysis of the same. In such cases, one common choice is a constant baseline hazard rate (Dellaportas and Smith, 1993; Ibrahim, Chen and Sinha, 2001) which amounts to an exponential baseline distribution. Typically, a conjugate gamma prior is assigned to the constant hazard rate. For the regression vector, on the other hand, an oft recommended prior is either the multivariate normal or uniform over the appropriate dimensional Euclidean space. While the former always ensures a proper posterior, the same cannot necessarily be said of the latter when the uniform distributions have infinite support. However, use of the latter is not uncommon (e.g. Dellaportas and Smith, 1993; Ibrahim, Chen and Sinha, 2001). Thus, it is of interest to find conditions under which the resulting posteriors are proper, so that inference based on them becomes meaningful. To our knowledge, this propriety issue has not been discussed elsewhere, in spite of its importance in Bayesian inference.

The paper addresses this problem in the special case of a simple linear regression, i.e. when the regression vector is one-dimensional. As stated in the preceding paragraph, the baseline distribution is assumed to be exponential. We have provided necessary and sufficient conditions for the propriety of posteriors in Section 2. Section 3 contains some concluding remarks.

2 The Main Result

Suppose we have data on n individuals, of which there are d distinct event times and $n-d$ right-censored survival times. We denote the event times by $t_1, t_2, \cdots, t_d$ and the survival times by $t_{d+1}, t_{d+2}, \cdots, t_n$. With an exponential baseline hazard distribution, the hazard rate $\lambda(t|x)$ in Cox's proportional hazard model, in its generic version, is given by

$$\lambda(t|x) = \lambda \exp(\beta x),\ t > 0, -\infty < \beta < \infty.$$

This leads to the likelihood function

$$L(\lambda, \beta) \propto \lambda^d \exp[\beta \sum_{i=1}^{d} x_i - \lambda \sum_{i=1}^{n} t_i \exp(\beta x_i)]. \tag{2.1}$$

Consider the prior $\pi(\lambda, \beta) \propto \exp(-a\lambda)\lambda^{b-1}$, where $a > 0$ and $b > 0$. Writing $\boldsymbol{t} = (t_1, \cdots, t_n)$, the joint posterior is given by

$$\pi(\lambda, \beta|\boldsymbol{t}) \propto \lambda^{b+d-1} \exp[\beta \sum_{i=1}^{d} x_i - \lambda(a + \sum_{i=1}^{n} t_i \exp(\beta x_i))].$$

Necessary and sufficient conditions for the propriety of this joint posterior is equivalent to the propriety of the marginal posterior of β, and the latter is given by

$$\pi(\beta|\boldsymbol{t}) \propto \exp(\beta \sum_{i=1}^{d} x_i)/[a + \sum_{i=1}^{n} t_i \exp(\beta x_i)]^{b+d}. \tag{2.2}$$

We now find necessary and sufficient conditions for the propriety of this posterior, i.e. conditions under which the integral (with respect to β) of the right hand side of (2.2) over $(-\infty, \infty)$ is finite. We begin with the case $d > 0$, i.e. not all observations are censored. Let $\bar{x}_d = d^{-1}\sum_{i=1}^{d} x_i$, $x_{\max} = \max(x_1, \cdots, x_n)$ and $x_{\min} = \min(x_1, \cdots, x_n)$. The finiteness, as stated above, will depend on the interrelationship between $\bar{x}_d$, $x_{\max}$ and $x_{\min}$.

To this end, first we note that the expression in the right hand side of (2.2) is bounded above and below by $\exp(\beta \sum_{i=1}^{d} x_i)/[a + t_{\min} \sum_{i=1}^{n} \exp(\beta x_i))]^{b+d}$ and $\exp(\beta \sum_{i=1}^{d} x_i)/[a + t_{\max} \sum_{i=1}^{n} \exp(\beta x_i))]^{b+d}$ respectively, where $t_{\min} = \min(t_1, \cdots, t_n)$ and $t_{\max} = \max(t_1, \cdots, t_n)$. Hence, it suffices to find necessary and sufficient conditions for finiteness of an integral of the form $\int_{-\infty}^{\infty} \exp(\beta \sum_{i=1}^{d} x_i)/[1 + c\sum_{i=1}^{n} \exp(\beta x_i))]^{b+d} d\beta$, $c > 0$. Further, with the transformation $u = \exp(\beta)$, this integral reduces to

$$\int_0^{\infty} u^{\sum_{i=1}^{d} x_i - 1}/[1 + c\sum_{i=1}^{n} u^{x_i}]^{b+d} du. \tag{2.3}$$

We find now necessary and sufficient conditions for finiteness of the integral given in (2.3). This requires consideration of several different cases. We begin with the situation when the x_i are not all equal, i.e. $x_{\max} - x_{\min} > 0$. The different cases are considered separately.

Case 1. $x_{\min} < x_{\max} \leq 0$. Let $A = 1 + c\sum_{i=1}^{n} u^{x_i}$. Then as $u \to 0+$, $A = \mathrm{O}(u^{x_{\min}})$, while as $u \to \infty$, $A = \mathrm{O}(1)$. Accordingly, $\int_0^1 [u^{\sum_{i=1}^{d} x_i - 1}/A^{b+d}] du = I_1$, (say), is finite if and only if (i) $d\bar{x}_d - (b+d)x_{\min} > 0$, while $\int_1^{\infty} [u^{\sum_{i=1}^{d} x_i - 1}/A^{b+d}] du = I_2$, (say), is finite if and only if (ii) $\bar{x}_d < 0$.

Note that (i) holds trivially in this case. Thus, (2.3) is finite if and only if (ii) holds.

Case 2a. $x_{\min} \le 0 < x_{\max}$. In this case, as $u \to 0+$, $A = \mathrm{O}(u^{x_{\min}})$, while if $u \to \infty$, $A = \mathrm{O}(u^{x_{\max}})$. Thus I_1 is finite if and only if (i) holds, while I_2 is finite if and only if (iii) $d\bar{x}_d - (b+d)x_{\max} < 0$ holds. Note that (iii) holds trivially in this case.

Case 2b. $x_{\min} < 0 \le x_{\max}$. The necessary and sufficient conditions for the finiteness of I_1 and I_2 are the same as in 2a. However, (i) holds trivially in this case.

Case 3. $0 \le x_{\min} < x_{\max}$. Here as $u \to 0+$, $A = \mathrm{O}(1)$, while as $u \to \infty$, $A = \mathrm{O}(u^{x_{\max}})$. Hence, I_1 is finite if and only if (iv) $\bar{x}_d > 0$, while I_2 is finite if and only if (iii) holds.

Next if all the x_i are equal to zero, then the integrand given in (2.3) is proportional to u^{-1}, which is not integrable over $(0, \infty)$ implying the impropriety of the posterior. If on the other hand all the x_i are equal to $x_0 (\neq 0)$, then the integrand in (2.3) reduces to $u^{dx_0-1}/[1+cnu^{x_0}]^{b+d}$. Since $b > 0$ and $d > 0$, the integral given in (2.3) is always finite.

Finally, when $d = 0$, i.e. all the observations are censored, the integrand in (2.3) reduces to $u^{-1}[1 + c\sum_{i=1}^{n} u^{x_i}]^{-b}$. Since $b > 0$, this integral is finite if and only if $x_{\min} < 0 < x_{\max}$.

3 Concluding Remarks

This note initiates the study of a topic of current interest. Further cases, both parametric and semiparametric, can be considered. For example, one can study propriety of posteriors with Weibull or lognormal baseline hazard distributions, or when the baseline hazard is generated from a Dirichlet process or a Gamma process with independent increments. Also of interest is the consideration of a general p-dimensional regression vector with infinite support.

Acknowledgements

Datta's research was partially supported by NSF Grants DMS 0071642 and SES-0241651, and NSA Grant MDA904-03-1-0016. Ghosh's research was partially supported by NSF Grant SES 9911485 and NIH Grant R01-85414. Mallick's research was partially supported by National Cancer Institute Grant CA-57030.

References

[1] Cox, D.R. (1972). Regression models and life tables. *Journal of the Royal Statistical Society, Series B*, **21**, 187-220.

[2] Dellaportas, P. and Smith, A.F.M. (1993). Bayesian inference for generalized linear and proportional hazards models via Gibbs sampling. *Applied Statistics*, **42**, 443-459.

[3] Ibrahim, J.G., Chen, M-H. and Sinha, D. (2001). *Bayesian Survival Analysis*. Springer, New York.