Journal of Statistical Research *ISSN 0256 - 422 X*
2003, Vol. 37, No. 1, pp. 21-29
Bangladesh

CONFIDENCE BOUNDS OF SCALE PARAMETERS USING PILOT SAMPLES

Paul Chiou
Department of Mathematics, Lamar University
Beaumont, Texas 77710-0047, USA
Email: chiou@math.lamar.edu

SUMMARY

The first sample in a possible double sampling plan can be referred to as a pilot sample. A pilot study is often done prior to a more thorough study to determine whether further sampling should be conducted. If the second sample is taken, then the data from these two samples can be used for inference purposes. This paper studies the confidence bounds of scale parameters using this double sampling procedure. The confidence bound obtained from the two samples constitutes an exact conditional confidence bound (CCB) for the parameter. The conditional coverage probability of the usual confidence bound (UCB) is obtained by computing the coverage probability under the conditional probability density function. It is shown that the conditional coverage probability of the UCB is uniformly less than the nominal level.

Keywords: Conditional confidence level; conditional distribution; confidence bound; exponential distribution; extreme-value distribution; normal distribution; semirelevant subset; type II censored sample.

AMS Classification: Primary 62F25; Secondary 62E15.

1 Introduction

Many studies use a pilot sample to determine whether a meaningful or useful effect is likely to be present in a population. For example, in a normal lifetime situation where the experimenter is interested in determining if the standard deviation is greater than one will be suitable for the model. Because conducting an experiment is expensive and entails considerable inconvenience, the experimenter is tempted to first take a pilot sample. On the basis of the pilot sample, he will decide whether the standard deviation is greater than one; if so, he then take an additional sample and construct a confidence bound using these two samples. If the pilot sample indicated otherwise, then it is best for all concerned not to take a second sample. In other words, after a pilot sample is taken, the null hypothesis that the standard deviation is equal to one against the alternative hypothesis that the standard deviation is greater than one will be tested. If the null hypothesis is rejected, then a more precise estimate of the parameter is desired and a second sample is called for. Based on the data from the two samples, an exact conditional confidence bound is constructed for the scale parameter. The construction of a confidence bound can be based on similar considerations if the experimenter is interested in whether the standard deviation is less than one.

Conditional inference can be traced back to Fisher (1956), Cox (1958), and others. The best-known study is probably that of Kiefer (1977). Other contributions have come from Brown (1967, 1978), Buehler (1959), Buehler and Feddersen (1963), Casella (1987), Chiou and Han (1995, 1999), Cox (1958, 1980), Fraser (1977), Olshen (1973), Robinson (1976, 1979), and Wallace (1959). In the next section we consider the statistical setting of the problem and derive the exact conditional confidence bound (CCB) for the parameter. The lower or upper confidence bound is obtained by solving an integral equation. The conditional confidence level of the usual confidence bound (UCB) is obtained by computing the coverage probability of an interval with UCB under the conditional probability density function (pdf). It is shown that the conditional confidence level of the UCB is uniformly less than the nominal level. In Section 3 a discussion for the UCB in terms of coverage probability is made. It is concluded that the UCB is not suitable when considered from a conditional viewpoint, and therefore a CCB can provide an alternative if a pilot sample is used prior to a more thorough study.

2 Confidence Bound

Let $X_1, \cdots, X_{n_1}$ be a pilot sample from a distribution $F(x;\mu,\theta)$, where μ is the location parameter and θ is the scale parameter, and $X_{n_1+1}, \cdots, X_n$ be the second sample from $F(x;\mu,\theta)$ if necessary. Without loss of generality, let $S_1(X_1, \cdots, X_{n_1})$ and $S_2(X_{n_1+1}, \cdots, X_n)$ be statistics for estimating θ, respectively. In addition, $S_i/\theta \sim \chi^2_{r_i}$, $i = 1, 2$, and that S_1 and S_2 are independent. Now if the null hypothesis of interest is $H_0 : \theta = \theta_0$ versus $H_a : \theta > \theta_0$, we consider the event

$$A = \{(x_1, \cdots, x_{n_1}) \, : \, S_1/\theta_0 \geq \chi^2_{r_1}(\alpha)\}, \tag{2.1}$$

where $\chi^2_{r_1}(\alpha)$ is the $100(1-\alpha)$ percentage point of the chi-squared distribution with r_1 degrees of freedom. When A occurs, i.e., $H_0 : \theta = \theta_0$ is rejected, take a second sample of size n_2. Based on $\mathbf{X} = (X_1, \cdots, X_{n_1}, X_{n_1+1}, \cdots, X_n), n = n_1 + n_2$, find a lower confidence bound $L(\mathbf{X})$ for θ such that

$$P_\theta\{L(\mathbf{X}) \leq \theta | A\} = 1 - p; \tag{2.2}$$

that is, such that the conditional probability of coverage is $(1-p)$. On the other hand, if the null hypothesis of interest is $H_0 : \theta = \theta_0$ versus $H_a : \theta < \theta_0$, we then consider the event

$$B = \{(x_1, \cdots, x_{n_1}) \, : \, S_1/\theta_0 \leq \chi^2_{r_1}(1-\alpha)\}, \tag{2.3}$$

where $\chi^2_{r_1}(1-\alpha)$ is the 100α percentage point of the chi-squared distribution with r_1 degrees of freedom. If B occurs, then a second sample is taken and an upper confidence bound $U(\mathbf{X})$

for θ such that

$$P_\theta\{U(\mathbf{X}) \ \geq \ \theta|B\} = 1 - p; \tag{2.4}$$

can be constructed based on $\mathbf{X}$.

Let $S = S_1 + S_2$. Since S_1 and S_2 are independent, $S/\theta \sim \chi^2_{r_1+r_2}$ and

$$f_\theta(s_1|s) = \frac{f_1(s_1)f_2(s_2)}{f(s)},$$

by letting $Y \ = \ S_1/S$, it follows that the conditional density function $f_\theta(y|s)$ is a beta distribution with parameters $(r_1/2,\, r_2/2)$, and $P\{A|s\}$ is in terms of the incomplete beta function. The conditional density function of S given A is

$$f_\theta(s|A) \ = \ f_\theta(s) \cdot P\{A|s\}/P_\theta(A), \tag{2.5}$$

while the conditional density function of S given B is

$$f_\theta(s|B) \ = \ f_\theta(s) \cdot P\{B|s\}/P_\theta(B), \tag{2.6}$$

where

$$f_\theta(s) \ = \ \left\{1/\left[\Gamma\left(\frac{r}{2}\right)(2\theta)^{r/2}\right]\right\} e^{-s/(2\theta)}\, s^{r/2-1}, r = r_1 + r_2, \tag{2.7}$$

$$P\{A|s\} \ = \ 1 \ - \ I\left[\theta_0\, \chi^2_{r_1}(\alpha)/s; \frac{r_1}{2}, \frac{r_2}{2}\right], \tag{2.8}$$

$$P\{B|s\} \ = \ I\left[\theta_0\, \chi^2_{r_1}(1-\alpha)/s; \frac{r_1}{2}, \frac{r_2}{2}\right], \tag{2.9}$$

$I[z; \cdot, \cdot]$ is the incomplete beta function,

$$P_\theta(A) \ = \ 1 \ - \ \Gamma\left[\theta_0\, \chi^2_{r_1}(\alpha)/\theta; \frac{r_1}{2}, 2\right], \tag{2.10}$$

$$P_\theta(B) \ = \ \Gamma\left[\theta_0\, \chi^2_{r_1}(1-\alpha)/\theta; \frac{r_1}{2}, 2\right], \tag{2.11}$$

and $\Gamma[z; \cdot, \cdot]$ is the incomplete gamma function. Thus, we have our main result in this section as follows.

Theorem 2.1 A conditional (on A) $100(1-p)\%$ lower confidence bound for θ can be obtained by solving the following integral equation in θ:

$$\int_{s^*}^{\infty} f_\theta(s|A)\, ds \ = \ p, \tag{2.12}$$

where s^* is the observed value of S, and $f_\theta(s|A)$ is given as (2.5). On the other hand, a conditional (on B) $100(1-p)\%$ upper confidence bound for θ can be obtained by solving the following integral equation in θ:

$$\int_{s^*}^{\infty} f_\theta(s|B)\, ds \ = \ 1 - p, \tag{2.13}$$

where $f_\theta(s|B)$ is given as (2.6).

Remark 2.1 Case (1): A confidence bound for the variance of the normal distribution is of interest. Let $S_1 = \sum_{j=1}^{n_1}(X_j - \bar{X}_1)^2$, where $\bar{X}_1$ is the sample mean of the pilot sample ($i = 1$), and $S_2 = \sum_{j=n_1+1}^{n}(X_j - \bar{X}_2)^2$, where $\bar{X}_2$ is the sample mean of the second sample ($i = 2$). Then S_i/θ, $i = 1, 2$, has a chi-squared distribution with $r_i = (n_i - 1)$ degrees of freedom. Case (2): A confidence bound for the exponential scale parameter is of interest. Let $X_{i,1}, \cdots, X_{i,m_i}$ be the type II censored sample from sample i, $i = 1, 2$, and $S_i = 2(\sum_{j=1}^{m_i}(X_{i,j} - X_{i,1}) + (n_i - m_i)(X_{i,m_i} - X_{i,1}))$. Then S_i/θ has a chi-squared distribution with $r_i = 2(m_i - 1)$ degrees of freedom. Case (3): A confidence bound for the extreme-value scale parameter is of interest. Let $X_{i,1}, \cdots, X_{i,m_i}$ be the type II censored sample from sample i, $i = 1, 2$, and $\hat{\theta}_i = \sum_{j=1}^{m_i} b_i(j; n_i, m_i)\, X_{i,j}$, where $b_i(j; n_i, m_i)$, $j = 1, \cdots, m_i$, is the coefficient of $X_{i,j}$ for the best linear unbiased estimator (BLUE) of θ (Lieblein and Zelen (1956)). The values of $b_i(j; n_i, m_i)$ are given in White (1964) and Nelson (1982). Then using the chi-squared approximation, $S_i/\theta = r_i\, \hat{\theta}_i/\theta$ has a chi-squared distribution with $r_i = 2/Var\{\hat{\theta}_i/\theta\}$ degrees of freedom (Engelhardt and Bain (1973); Lawless and Mann (1976)). Note that (S_i/r_i), $i = 1, 2$, is the minimum variance unbiased estimator (MVUE) for θ in Cases (1) and (2), and it is the BLUE for θ in Case (3) using the pilot sample ($i = 1$) and the second sample ($i = 2$), respectively.

For given values of θ_0, α, p and s^*, a lower bound $\hat{\theta}_L^c = L(\mathbf{X})$ or an upper bound $\hat{\theta}_U^c = U(\mathbf{X})$ for the CCB can be obtained by simply solving equation (2.12) or (2.13) for θ. In an attempt to solve equation (2.12) numerically in the case of extreme-value scale parameter for comparison, we utilize the NAG (1993) routines D01AJF, G01EEF, G01EFF, G01FCF and S14AAF in conjunction with a Fortran program. Table 1 presents values of the unbiased estimator $\hat{\theta}$ defined by S/r, values of the UCB $\hat{\theta}_L$ defined by $S/\chi_r^2(p)$, and values of the CCB $\hat{\theta}_L^c$ for $\theta_0 = 1$, $\alpha = .05, p = .05, n_1 = 4, m_1 = 2, n_2 = 8, m_2 = 6$, and various values of s^*. Note that $r_1 = 2.307$ when $n_1 = 4, m_1 = 2$, and $r_2 = 12.903$ when $n_2 = 8, m_2 = 6$ (Engelhardt and Bain (1973); Lawless and Mann (1976)).

Table 1: Values of $\hat{\theta}$, $\hat{\theta}_L$, $\hat{\theta}_L^c$ for $\theta_0 = 1$, $\alpha = .05$, $p = .05, n_1 = 4, m_1 = 2, n_2 = 8, m_2 = 6$

s^*	$\hat{\theta}$	$\hat{\theta}_L$	$\hat{\theta}_L^c$	s^*	$\hat{\theta}$	$\hat{\theta}_L$	$\hat{\theta}_L^c$
12	0.7890	0.4749	0.2179	32	2.1039	1.2663	1.0184
16	1.0519	0.6332	0.3783	36	2.3669	1.4246	1.1781
20	1.3149	0.7915	0.5385	40	2.6298	1.5829	1.3378
24	1.5779	0.9498	0.6986	50	3.2873	1.9787	1.7366
28	1.8409	1.1080	0.8585	60	3.9448	2.3744	2.1351

To study the actual coverage probability that is provided at the nominal $100(1-p)\%$

level, we compute the coverage probability of the following interval with UCB based on $\mathbf{X}$,

$$\Omega(\mathbf{X}) : \hat{\theta}_L = (S/\chi_r^2(p)) \leq \theta, \tag{2.14}$$

under the conditional pdf $f_\theta(s|A)$. It leads to Theorem 2.2 and subsequently Theorem 2.3 as follows.

Theorem 2.2 The conditional confidence level of the UCB $\hat{\theta}_L$, i.e., $\int_\Omega f_\theta(s|A)ds$, is

$$CCP(\theta,\alpha) = \int_{\theta_0\,\chi_{r_1}^2(\alpha)}^{\theta\,\chi_r^2(p)} f_\theta(s|A)ds, \quad \text{if} \quad \theta \geq (\theta_0\,\chi_{r_1}^2(\alpha)/\chi_r^2(p))$$

$$\text{and} \quad CCP(\theta,\alpha) = 0, \quad \text{otherwise}, \tag{2.15}$$

where $\chi_\nu^2(\gamma)$ is the $100(1-\gamma)$ percentage point of the chi-squared distribution with ν degrees of freedom.

Proof. Follows from a simple argument using (2.1) and (2.14).

Theorem 2.3 The $CCP(\theta,\alpha)$ of the UCB $\hat{\theta}_L$ is a nondecreasing function of θ ($\theta > 0$), and $CCP(\theta,\alpha) \leq (1-p)$, where $(1-p)$ is the nominal level.

Proof. Note that $\theta > 0$ and $\chi_r^2(p) > 0$. Since

$$\frac{\partial\{CCP(\theta,\alpha)\}}{\partial\theta} = 0, \quad \text{if} \quad 0 < \theta < (\theta_0\,\chi_{r_1}^2(\alpha)/\chi_r^2(p))$$

$$\text{and} \quad \frac{\partial\{CCP(\theta,\alpha)\}}{\partial\theta} = f_\theta(\theta\,\chi_r^2(p)|A)\,\chi_r^2(p) > 0, \quad \text{if} \quad \theta \geq (\theta_0\,\chi_{r_1}^2(\alpha)/\chi_r^2(p)),$$

it follows that $CCP(\theta,\alpha)$ is a nondecreasing function. To prove the second part of the Theorem, first note that $CCP(\theta,\alpha) = 0$ if $0 < \theta < (\theta_0\,\chi_{r_1}^2(\alpha)/\chi_r^2(p))$. If $\theta \geq (\theta_0\,\chi_{r_1}^2(\alpha)/\chi_r^2(p))$,

$$\begin{aligned} CCP(\theta,\alpha) &= \int_{\theta_0\,\chi_{r_1}^2(\alpha)}^{\theta\,\chi_r^2(p)} f_\theta(s|A)ds \\ &= \int_{\theta_0\,\chi_{r_1}^2(\alpha)}^{\theta\,\chi_r^2(p)} (f_\theta(s)\cdot P\{A|s\}/P_\theta(A))ds \\ &= \int_{\theta_0\,\chi_{r_1}^2(\alpha)/\theta}^{\chi_r^2(p)} (f(v)\cdot P\{A|\theta\,v\}/P_\theta(A))dv, \end{aligned}$$

where $f(v)$ is the pdf of the chi-squared distribution with r degrees of freedom. Since $(\theta_0\,\chi_{r_1}^2(\alpha)/\theta) \to 0$, $P\{A|\theta\,v\} \to 1$, and $P_\theta(A) \to 1$ as $\theta \to \infty$, it follows that

$$CCP(\theta,\alpha) \to \int_0^{\chi_r^2(p)} f(v)dv = (1-p) \quad \text{as } \theta \to \infty.$$

This completes the proof.

Remark 2.2 Note that $CCP(\theta,\alpha) = P_\theta\{\theta \in \Omega(\mathbf{X})|A\} = P_\theta\{S/\chi_r^2(p) \leq \theta|A\}$. From Theorem 2.3 we have that $CCP(\theta,\alpha)$ is always less than $(1-p)$, and tends to $(1-p)$ as

$\theta \to \infty$. Therefore, the set $C = \{(x_1, \cdots, x_n) : S_1(x_1, \cdots, x_{n_1})/\theta_0 \geq \chi^2_{r_1}(\alpha)\}$ is said to be a negatively biased semirelevant subset for the UCB with unconditional confidence level $(1-p)$ (see Lehmann (1986), p. 554). In other words, $CCP(\theta, \alpha) = P_\theta\{\theta \in \Omega(\mathbf{X}) | \mathbf{X} \in C\} \leq (1-p)$ for all θ with strict inequality holding for at least some θ.

For given values of θ/θ_0, α and p the conditional coverage probability of the UCB can be computed by utilizing the NAG (1993) routines aforementioned in conjunction with a Fortran program. Table 2 presents the true (conditional) coverage probability $CCP(\theta, \alpha)$ in the case of extreme-value scale parameter for $p = .05, n_1 = 4, m_1 = 2, n_2 = 8, m_2 = 6$, and $\alpha = .01, .025, .05, .10$. The $CCP(\theta, \alpha)$ of the UCB $\hat{\theta}_L$ is a nondecreasing function of θ. For values of θ/θ_0 close to zero, the conditional coverage probability is far less than the nominal 0.95. As θ/θ_0 increases, the $CCP(\theta, \alpha)$ converges to the nominal 0.95 from below. On the other hand, for fixed p the larger the level of significance α, the closer the conditional probability is to the nominal level.

3 Discussion

Cohen and Sackrowitz (1996) have proposed the choices for n_1 and n_2 in the case of a normal mean. The proposition can be applied to the choices for n_1 and n_2 in the case of the normal variance. The choices for n_1 and n_2 are dictated by the practical considerations of the underlying study. To determine n_1 in the censored samples case of scale parameters, one must specify α, m_1 and perhaps indicate a value of θ and a desirable power at the specified value of θ/θ_0 for the initial chi-squared test. The determination of n_2 can be based on similar considerations of the underlying study.

From Table 1 we see that $\hat{\theta}^c_L$ tends to be closer to 0 than $\hat{\theta}_L$ if s^* is small. As s^* gets larger, $\hat{\theta}^c_L$ becomes and resembles $\hat{\theta}_L$ which is consistent with the fact that the conditional pdf $f_\theta(s|A)$ converges to the unconditional pdf $f_\theta(s)$ as $\theta \to \infty$. For example, when $\theta_0 = 1$, $\alpha = .05$ and $s^* = 20$, the UCB is 0.7915 while the CCB is 0.5385 for $r_1 = 2.307$ $(n_1 = 4, m_1 = 2)$ and $r_2 = 12.903$ $(n_2 = 8, m_2 = 6)$. However, when $s^* = 60$, the UCB is 2.3744 and the CCB is 2.1351. From Table 2 we see that the coverage probability of the UCB $\hat{\theta}_L$ is less than the nominal level which is consistent with what Theorem 2.3 states. For values of θ/θ_0 over an interval around one, the coverage probability is substantially less than the nominal level unless α is large. For example, when $\alpha = .05$, the conditional coverage probability of a nominal 95% UCB for $r_1 = 2.307$ $(n_1 = 4, m_1 = 2)$ and $r_2 = 12.903$ $(n_2 = 8, m_2 = 6)$ is less than 90.5% if $0 < \theta/\theta_0 < 2.4$, and increases to reach 94.8% as $\theta/\theta_0 = 30$. In addition, the conditional coverage probability of the UCB $\hat{\theta}_L$ converges to $(1-p)$ as $\theta \to \infty$ or $\alpha \to 1$.

Evidently, θ is unknown in practice, so the UCB is not appropriate if a pilot sample encourages further investigation unless a large value of α is used for the pre-test. On the other hand, if a small value of α is used for the pre-test as in many practical situations, then the appropriate confidence bound is the CCB not the UCB because the coverage probability of the UCB $\hat{\theta}_L$ is uniformly less than the nominal level and could be far less than the nominal level over an interval of θ/θ_0 around $\theta/\theta_0 = 1$.

Table 2: $CCP(\theta, \alpha)$ of 95% nominal UCB for $n_1 = 4, m_1 = 2, n_2 = 8, m_2 = 6$

	α					α			
θ/θ_0	.01	.025	.05	.10	θ/θ_0	.01	.025	.05	.10
0.1	0.000	0.000	0.000	0.000	2.6	0.877	0.897	0.909	0.921
0.2	0.000	0.000	0.000	0.000	2.8	0.885	0.902	0.913	0.923
0.3	0.000	0.000	0.001	0.086	3.0	0.891	0.906	0.916	0.926
0.4	0.000	0.011	0.117	0.360	3.2	0.896	0.910	0.919	0.928
0.5	0.013	0.139	0.335	0.553	3.4	0.901	0.913	0.922	0.929
0.6	0.114	0.319	0.502	0.667	3.6	0.904	0.916	0.924	0.931
0.7	0.261	0.466	0.613	0.736	3.8	0.908	0.918	0.925	0.932
0.8	0.394	0.571	0.687	0.781	4.0	0.911	0.920	0.927	0.933
0.9	0.499	0.645	0.738	0.812	4.5	0.916	0.924	0.930	0.935
1.0	0.579	0.699	0.774	0.834	5.0	0.921	0.928	0.933	0.937
1.1	0.639	0.739	0.800	0.851	5.5	0.924	0.930	0.934	0.939
1.2	0.685	0.769	0.821	0.863	6.0	0.927	0.932	0.936	0.940
1.3	0.721	0.792	0.837	0.873	7.0	0.931	0.935	0.938	0.941
1.4	0.750	0.811	0.849	0.881	8.0	0.934	0.937	0.940	0.943
1.5	0.773	0.826	0.860	0.888	9.0	0.936	0.939	0.941	0.944
1.6	0.791	0.839	0.868	0.893	10	0.938	0.940	0.942	0.944
1.7	0.807	0.849	0.875	0.898	11	0.939	0.941	0.943	0.945
1.8	0.820	0.858	0.882	0.902	12	0.940	0.942	0.944	0.945
1.9	0.831	0.865	0.887	0.906	15	0.942	0.944	0.945	0.946
2.0	0.841	0.871	0.891	0.909	20	0.945	0.946	0.947	0.947
2.2	0.856	0.882	0.899	0.914	30	0.947	0.947	0.948	0.948
2.4	0.868	0.890	0.905	0.918	40	0.948	0.948	0.948	0.949

References

[1] Brown, L. D. (1967). The conditional level of student's t-test. *Annals of Mathematical Statistics*, **38**, 1068-1071.

[2] Brown, L. D. (1978). A contribution to Kiefer's theory of conditional confidence procedures. *Annals of Statistics*, **6**, 59-71.

[3] Buehler, R. J. (1959). Some validity criteria for statistical inferences. *Annals of Mathematical Statistics*, **30**, 845-863.

[4] Buehler, R. J. and Feddersen, A. P. (1963). Note on a conditional property of student's *t*. *Annals of Mathematical Statistics*, **34**, 1098-1100.

[5] Casella, G. (1987). Conditionally acceptable recentered set estimators. *Annals of Statistics*, **15**, 1363-1371.

[6] Chiou, P. and Han, C.-P. (1995). Conditional interval estimation of the exponential location parameter following rejection of a pre-test. *Communications in Statistics-Theory and Methods*, **24**, 1481-1492.

[7] Chiou, P. and Han, C.-P. (1999). Conditional interval estimation of the ratio of variance components following rejection of a pre-test. *Journal of Statistical Computation and Simulation*, **63**, 105-119.

[8] Cohen, A. and Sackrowitz, H. B. (1996). Lower confidence bounds using pilot samples with an application to auditing. *Journal of the American Statistical Association*, **91**, 338-342.

[9] Cox, D. R. (1958). Some problems connected with statistical inference. *Annals of Mathematical Statistics*, **29**, 357-372.

[10] Cox, D. R. (1980). Local ancillarity. *Biometrika*, **67**, 279-286.

[11] Engelhardt, M. and Bain, L. J. (1973). Some complete and censored sampling results for the Weibull or extreme-value distribution. *Technometrics*, **15**, 541-549.

[12] Fisher, R. A. (1956). *Statistical Methods and Scientific Inference*, Edinburgh: Oliver and Boyd.

[13] Fraser, D. (1977). Confidence, posterior probability and the Buehler example. *Annals of Statistics*, **5**, 892-898.

[14] Kiefer, J. (1977). Conditional confidence statements and confidence estimators. *Journal of the American Statistical Association*, **72**, 789-827.

[15] Lawless, J. F. and Mann, N. R. (1976). Tests for homogeneity of extreme-value scale parameters. *Communications in Statistics*, **A5**, 389-405.

[16] Lehmann, E. L. (1986). *Testing Statistical Hypotheses* (2nd ed.), New York: John Wiley.

[17] Lieblein, J. and Zelen, M. (1956). Statistical investigation of the fatigue life of deep-groove ball bearings. *Journal of Research NBS*, **57**, 273-316.

[18] NAG (1993). *The NAG Fortran Library*, Oxford: The Numerical Algorithms Group Limited.

[19] Nelson, W. (1982). *Applied Life Data Analysis*, New York: John Wiley and Sons.

[20] Olshen, R. A. (1973). The conditional level of the F-test. *Journal of the American Statistical Association*, **68**, 692-698.

[21] Robinson, G. K. (1976). Properties of student's t and of the Behrens-Fisher solution to the two means problem. *Annals of Statistics*, **4**, 963-971.

[22] Robinson, G. K. (1979). Conditional properties of statistical procedures. *Annals of Statistics*, **7**, 742-755.

[23] Wallace, D. L. (1959). Conditional confidence level properties. *Annals of Mathematical Statistics*, **30**, 864-876.

[24] White, J. S. (1964). Least squares unbiased censored linear estimation for the log Weibull (extreme-value) distribution. *Industrial Mathematics*, **14**, 21-60.